A function \( f(x, y) \) is said to be homogeneous of degree \( n \) if \( f(tx, ty) = t^n f(x, y) \) for all \( t \). Here \( n \) is a positive integer. Homogeneous functions are very important in the study of elliptic curves and cryptography.

1. Show that the function \( r(x, y) = 4xy^6 - 2x^3y^4 + x^7 \) is homogeneous of degree 7.

\[
r(tx, ty) = 4txt^6 y^6 - 2t^3x^3t^4y^4 + t^7 x^7 = t^7 r(x, y).
\]

2. Give a nontrivial example of a function \( g(x, y) \) which is homogeneous of degree 9.

Answers will vary.

3. Show that if \( f(x, y) \) is homogeneous of degree \( n \) and sufficiently differentiable, then \( f(x, y) \) satisfies the equation

\[
x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf(x, y).
\]

(Hint: let \( X = tx \) and \( Y = ty \) and take the derivative with respect to \( t \) of the equation \( f(tx, ty) = t^n f(x, y) \) and consider the case of \( t = 1 \).)

Using the hint, we have

\[
\frac{\partial}{\partial t} f(X, Y) = \frac{\partial}{\partial t} t^n f(x, y),
\]

where

\[
\frac{\partial}{\partial t} f(X, Y) = \frac{\partial f(X, Y)}{\partial X} \frac{\partial X}{\partial t} + \frac{\partial f(X, Y)}{\partial Y} \frac{\partial Y}{\partial t} = \frac{\partial f(X, Y)}{\partial X} x + \frac{\partial f(X, Y)}{\partial Y} y
\]

and

\[
\frac{\partial}{\partial t} t^n f(x, y) = nt^{n-1} f(x, y).
\]

Setting these two equal gives

\[
\frac{\partial f(X, Y)}{\partial X} x + \frac{\partial f(X, Y)}{\partial Y} y = nt^{n-1} f(x, y).
\]

Notice that if \( t = 1 \), then \( X = x \) and \( Y = y \), therefore

\[
\frac{\partial f(x, y)}{\partial x} x + \frac{\partial f(x, y)}{\partial y} y = n^{n-1} f(x, y).
\]

4. Using the assumptions in problem 3, show that \( f(x, y) \) also satisfies

\[
x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial x^2} = n(n-1) f(x, y).
\]

Here we simply take another time derivative:

\[
\frac{\partial}{\partial t} \left( \frac{\partial f(X, Y)}{\partial X} x + \frac{\partial f(X, Y)}{\partial Y} y \right) = \frac{\partial}{\partial t} nt^{n-1} f(x, y),
\]

where

\[
\frac{\partial}{\partial t} \left( x \frac{\partial f(X, Y)}{\partial X} \right) = x \left[ \frac{\partial^2 f(X, Y)}{\partial X^2} \frac{\partial X}{\partial t} + \frac{\partial^2 f(X, Y)}{\partial X \partial Y} \frac{\partial Y}{\partial t} \right] = x \left[ \frac{\partial^2 f(X, Y)}{\partial X^2} x + \frac{\partial^2 f(X, Y)}{\partial X \partial Y} y \right] = x^2 \frac{\partial^2 f(X, Y)}{\partial X^2} + xy \frac{\partial^2 f(X, Y)}{\partial X \partial Y}.
\]
Similarly,
\[
\frac{\partial}{\partial t} \left( y \frac{\partial f(X, Y)}{\partial Y} \right) = y \left[ \frac{\partial^2 f(X, Y)}{\partial Y^2} \frac{\partial Y}{\partial t} + \frac{\partial^2 f(X, Y)}{\partial Y \partial X} \frac{\partial X}{\partial t} \right] \\
= y \left[ \frac{\partial^2 f(X, Y)}{\partial Y^2} y + \frac{\partial^2 f(X, Y)}{\partial Y \partial X} x \right] \\
= y^2 \frac{\partial^2 f(X, Y)}{\partial Y^2} + xy \frac{\partial^2 f(X, Y)}{\partial Y \partial X}.
\]

But notice that
\[
\frac{\partial^2 f(X, Y)}{\partial X \partial Y} = \frac{\partial^2 f(X, Y)}{\partial Y \partial X},
\]

therefore,
\[
\frac{\partial}{\partial t} \left( \frac{\partial f(X, Y)}{\partial X} x + \frac{\partial f(X, Y)}{\partial Y} y \right) = x^2 \frac{\partial^2 f(X, Y)}{\partial X^2} + 2xy \frac{\partial^2 f(X, Y)}{\partial X \partial Y} + y^2 \frac{\partial^2 f(X, Y)}{\partial Y^2}.
\]

Next, the left hand side:
\[
\frac{\partial}{\partial t} n t^{n-1} f(x, y) = n(n-1)t^{n-2} f(x, y).
\]

Setting these equal we have
\[
x^2 \frac{\partial^2 f(X, Y)}{\partial X^2} + 2xy \frac{\partial^2 f(X, Y)}{\partial X \partial Y} + y^2 \frac{\partial^2 f(X, Y)}{\partial Y^2} = n(n-1)t^{n-2} f(x, y).
\]

Once again, when \( t = 1 \), one arrives at
\[
x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y).
\]